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Interpolation of Cones and Shape-Preserving Approximation

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In this paper, we present an approach to shape-preserving approximation based on interpolation space theory. In particular, we prove the corresponding approximation result related to the intersection property of the cone of nonnegative functions with respect to the couple $(L_p, B_p^{\alpha,\infty})$. © 2002 Elsevier Science (USA)

Key Words: K-functional of the couple of cones; interpolation cone of K-method; one-sided intersection property; shape-preserving approximation.

1. INTRODUCTION

It has been known that the quantitative approximation theory is closely connected with the real interpolation method. This connection has been investigated by Peetre and Sparr [24] within the framework of their interpolation theory of abelian groups, and Brudnyi [4] and by Brudnyi and Krugljak [6], [7] using the concept of an "approximation family". Some new results and applications in the latter direction were presented in the papers of Pietsch [25], and DeVore and Popov [11]. In this paper, we would like to present an interpolation space approach to a new area of quantitative approximation theory, namely, to *approximation with constraints*, see e.g. [10, 12, 15, 16, 22, 27, 30, 31]. It is worth noting that these and other papers on this subject considered only univariate functions. We hope that the approach presented here might be useful in many applications for the multivariate case. We will choose a relatively simple (but highly nontrivial) approximation model of such a kind (see Theorem B) to show how results of interpolation theory for operators preserving a convex cone structure can work in this situation.

In order to formulate the main results we need a few basic definitions and the corresponding preliminary results.

Definition 1.1. An *approximation family* $\mathscr{A} = \{A_n\}_{n=1}^{\infty}$ is a collection of subsets A_n of a Banach space X satisfying the conditions:

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1. $A_n + A_m \subset A_{m+n}$, $n, m \in \mathbb{N}$;
2. $\lambda A_n \subset A_n$, $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$

(in particular, $0 \in A_n$ and $A_n \subset A_{n+1}$).

We put $A_0 := \{0\}$ and $A_\infty := X$, and denote the sequence $\{A_n;\ n \in \mathbb{Z}_+ \bigcup \{+\infty\}\}$ by $\mathscr{A}$.

Suppose now that Q is a convex cone in Banach space X. For every $x \in X \cap Q$ we set

$$e^Q_{\mathscr{A}_n}(x)_X := \inf\{\|x - a\|_X,\ a \in A_n \cap Q\}$$

and define the *approximation cone* $E^Q_{\alpha,p}(\mathscr{A})$ to be the set of elements $x \in X \cap Q$ satisfying

$$\|x\|_{E^Q_{\alpha,p}(\mathscr{A})} := \left\{ \sum_{n=0}^{\infty} (n+1)^{-1}((n+1)^\alpha e^Q_{\mathscr{A}_n}(x)_X)^p \right\}^{1/p} < \infty. \tag{1.1}$$

Here $1 \leqslant p \leqslant \infty$ and $0 < \alpha < \infty$. Quantity (1.1) defines a quasinorm on the cone $E^Q_{\alpha,p}(\mathscr{A})$.

Let Y be a linear space equipped with a semi-norm $|\cdot|_Y$, linearly embedded in X.

Definition 1.2 (DeVore and Lorentz [9]). (a) An approximation family $\mathscr{A} \cap Q$ satisfies *the abstract Jackson inequality* with respect to Y if

$$e^Q_{\mathscr{A}_n}(x)_X \leqslant Cn^{-r}|x|_Y, \qquad x \in Y \cap Q \tag{1.2}$$

is valid for all $n = 1, 2, \ldots$.

(b) $\mathscr{A}$ satisfies *the abstract Bernstein inequality* with respect to Y if

$$|\varphi|_Y \leqslant Cn^r\|\varphi\|_X, \qquad \varphi \in A_n \tag{1.3}$$

is valid for $n = 1, 2, \ldots$.

Here $r > 0$ is a fixed number and $C_1, C_2 > 0$ are absolute constants.

Our next definition introduces the *interpolation cone* $(X, Y \cap Q)_{\theta q}$, $\theta \in (0,1)$, $q \in [1, \infty]$ as the subset of elements $x \in X \cap Q$ satisfying

$$\|x\|_{(X,Y\cap Q)_{\theta q}} := \|K(x, \cdot; X, Y \cap Q)\|_{L^\theta_q(dt/t)} < \infty,$$

where

$$K(x, t; X, Y \cap Q) := \inf_{y \in Q \cap Y} \{\|x - y\|_X + t\|y\|_Y\}\ (t > 0)$$

and

$$\|f\|_{L_q^{\theta}(dt/t)} := \left(\int_0^{\infty} \left| \frac{f(t)}{t^{\theta}} \right|^q \frac{dt}{t} \right)^{1/q}.$$

It may be noted that K-functionals with cone constraints of a different kind were introduced in [18, 19, 26, 28, 29].

Remark 1.3. The same definitions of the interpolation cone $(X_0, X_1 \cap Q)_{\theta q}$ and $K(x, \cdot; X_0, X_1 \cap Q)$ can be related to the case of a Banach couple $\bar{X} = (X_0, X_1)$ and a convex cone $Q \subseteq X_0 + X_1$.

The following simple result can be derived in the precisely same manner as Theorem 9.3 of [9].

PROPOSITION 1.4. (a) *Suppose that condition* (1.2) *holds. Then for every* $x \in X \cap Q$ *and* $n = 1, 2, \ldots$

$$e^{Q}_{\mathscr{A}_n}(x)_X \leqslant (C+1)K(x, n^{-r}, X, Y \cap Q), \qquad n \in \mathbb{N}. \tag{1.4}$$

Here $C > 0$ *is an absolute constant.*

(b) *If, in addition, the condition* (1.3) *also holds, then*

$$E^{Q}_{\alpha,q}(\mathscr{A}) = (X, Y \cap Q)_{\alpha/r,q} \tag{1.5}$$

with equivalence of their (*quasi*) *norms.*

We now introduce the notion of *intersection properties* which allows to reduce the real interpolation for cones to the classical real interpolation, see, e.g. [20, 21, 28, 29].

Let $\bar{X} = (X_0, X_1)$ and Q be as in Remark 1.3.

DEFINITION 1.5. Q has the *right intersection property* (IP_+) with respect to $\bar{X}$ if

$$(X_0 + tX_1) \cap Q \subset X_0 + t(X_1 \cap Q) \tag{1.6}$$

with an embedding constant independent of t.

In other words, the IP_+ is equivalent to the two-sided inequality

$$K(x, t; X_0, X_1 \cap Q) \approx K(x, t; \bar{X}) \qquad (t > 0, x \in Q) \tag{1.7}$$

with constants of equivalence independent of x and t.

Since the right-hand side is evidently dominated by the left one, the main point is to prove the inequality

$$K(x,t;X_0,X_1\cap Q)\leqslant cK(t,x,\bar{X}),\qquad (f\in Q,\ t>0) \tag{1.8}$$

with c independent of x and t.

The IP_+ implies the following isomorphism of the cones:

$$Q\cap(X_0,X_1\cap Q)_{\theta q}=\bar{X}_{\theta q}\cap Q\qquad (0<\theta<1, 1\leqslant q\leqslant\infty). \tag{1.9}$$

DEFINITION 1.6. Q has the *weak right intersection property* (WIP_+) with respect to $\bar{X}$ if isomorphism (1.9) holds for every $\theta\in(0,1)$, $q\in[1,\infty]$.

For the results related to shape-preserving approximation we also need

DEFINITION 1.7. A cone Q is said to have the *restricted* WIP_+ with respect to $\bar{X}$ for a fixed $\theta\in(0,1)$, if the isomorphism

$$Q\cap(X_0,X_1\cap Q)_{\theta q}=\bar{X}_{\theta q}\cap Q \tag{1.10}$$

holds for every $q\in[1,\infty]$.

The next result shows the role of the intersection properties for approximation with constraints.

PROPOSITION 1.8. (a) *Suppose that condition* (1.2) *holds. Let* Q, *in addition, have the* IP_+ *with respect to* (X,Y). *Then for every* $x\in X\cap Q$ *and* $n\in\mathbb{N}$

$$e^{Q}_{\mathscr{A}_n}(x)_X\leqslant CK(x,n^{-r},X,Y). \tag{1.11}$$

Here $C>0$ *is an absolute constant.*

(b) *Suppose that conditions* (1.2) and (1.3) *hold. Let, in addition,* Q *have the restricted* WIP_+ *with respect to* (X,Y) *for* $\theta:=\frac{\alpha}{r}$. *Then*

$$E^{Q}_{\alpha,q}(\mathscr{A})=E_{\alpha,q}(\mathscr{A})\cap Q.$$

Here $0<\alpha<r$ *and* $0<q\leqslant\infty$.

Proof. (a) The result immediately follows from Proposition 1.4(a) and the IP_+.

(b) From Proposition 1.4(b) and the restricted WIP_+ with $\theta:=\frac{\alpha}{r}$ we have

$$E^{Q}_{\alpha,q}(\mathscr{A})=(X,Y\cap Q)_{\frac{\alpha}{r},q}=(X,Y)_{\frac{\alpha}{r},q}\cap Q.$$

But $(X,Y)_{\frac{\alpha}{r},q}=E_{\alpha,q}(\mathscr{A})$, see e.g. [9], and the proof is complete. ∎

In order to formulate the basic result of this paper, we introduce:

DEFINITION 1.9. For $\alpha > 0$ we write $r = [\alpha] + 1$. The semi-normed *Besov space* $\dot{B}_p^{\alpha,\infty}([a,b])$, $1 \leqslant p \leqslant \infty$ consists of all functions $f \in L_p[a,b]$ for which the semi-norm

$$|f|_{\dot{B}_p^{\alpha,\infty}} := \sup_{t>0} (t^{-\alpha}\omega_r(f,t)_{L_p[a,b]})$$

is finite.

The normed *Besov space* $B_p^{\alpha,\infty}([a,b])$ by definition equals $L_p \bigcap \dot{B}_p^{\alpha,\infty}$. Here ω_r denotes the modulus of smoothness of order r.

$\dot{W}_p^k := \dot{W}_p^k[0,1]$ denotes also the Sobolev space of functions $f \in L_p[0,1]$ equipped by the semi-norm $|f^{(k)}|_{L_p(0,1)}$.

Let now $M_0 := M_0[0,1]$ be the cone of nonnegative functions on $[0,1]$.

THEOREM A. *Let* $1 \leqslant p \leqslant \infty$ *and* $\alpha > 1/p$. *Then*

(i) M_0 *has the* IP_+ *with respect to the couple* $(L_p, \dot{W}_p^1)$.

(ii) M_0 *has the restricted* WIP_+ *with respect to the couple* $(L_p, \dot{B}_p^{\alpha,\infty})$ *for each* $\theta > \frac{1}{p\alpha}$, *see Definition* 1.7.

These interpolation results allow us to demonstrate a new approach to the problem of shape-preserving approximation. We choose for this goal a result of positive spline approximation (see Theorem B). Another approach of this type is seen in [15, 16]. In what follows, we require the following notions.

Let π_n be a subdivision of $[0,1)$ into the n half-open from the right intervals which are obtained by the following procedure. Write n in the form $n = 2^k + j$, where $n/2 < 2^k \leqslant n$ and $0 \leqslant j < 2^k$ (the representation is, clearly, unique). Divide $[0,1]$ into 2^k equal intervals and then divide each of the first j intervals from the left in half. The subdivision obtained will be denoted by $\pi_n := \{\Delta_{i,n}\}_{i=1}^n$.

DEFINITION 1.10. A function $s : [0,1] \to \mathbb{R}$ is said to be a spline of order k and smoothness l, if $s \in C^l[0,1]$ and the restriction of s to each interval $\Delta_{i,n}$ of the subdivision π_n is a polynomial of degree not exceeding $k-1$. We denote the linear space of such splines by $S_n^{k,l}[0,1]$.

When $k \geqslant 2$ we shall always assume that $0 \leqslant l \leqslant k-2$. (Otherwise $S_n^{k,l}$ is simply $\mathscr{P}_{k-1}$.) We shall also consider the case of $k=1$, i.e. the space of step functions which are constant on each of the intervals $\Delta_{i,n}$, $i = 1, 2, \ldots, n$. We let $l = -1$ in this case (more generally, $S_n^{k,-1}$ is the linear space of (possibly discontinuous) functions whose restrictions to each interval $\Delta_{i,n}$ of the subdivision π_n are polynomials of degree not exceeding $k-1$). Let

$$\sigma^{M_0}_{n,k,l}(f;L_p) := \inf_{s_n \in S^{k,l}_n \cap M_0} \|f - s_n\|_{L_p(0,1)}.$$

In the case of splines of maximal smoothness, i.e., $l = k-2$, we will omit the index l in all the preceding notation.

THEOREM B. (a) *Let* $p \in [1, \infty]$. *If* $f \in L_p(0,1) \cap M_0$ *then*

$$\sigma^{M_0}_{n,1}(f;L_p) \leqslant C\omega_1\left(f, \frac{1}{n}\right)_{L_p}, \qquad n \in \mathbb{N}, \tag{1.12}$$

where C *is a constant independent of* n.

(b) *The function* f *belongs to* $B^{\alpha,\infty}_p \cap M_0$, $\alpha > \frac{1}{p}$, if and only if there is a sequence $\{S_n \in S^{k,l}_n[0,1] \cap M_0\}_{n=1}^{\infty}$ of splines of degree $k = 3r+4$ and smoothness $l = r+1$, where r is the smallest integer $> \alpha$, such that

$$\sup_{n \in \mathbb{N}} n^{\alpha} \|f - S_n\|_{L_p(0,1)} < \infty. \tag{1.13}$$

Remark. (i) Taking into account part (a) of the theorem, we see that actually, for all α in the larger range $\alpha > 0$, except when $\alpha = p = 1$, the condition $f \in B^{\alpha,\infty}_p \cap M_0$ always implies (1.13).

(ii) Part (a) of this theorem is a special case of Theorem 4 in [15], formulated in this paper without proof. Apparently, statement (b) may also be proved by methods of paper [16], see proof of Theorem 15 therein.

2. PROOF OF THEOREM A

(i) Let $f \in L_p$ and $f \geqslant 0$. Set for $0 \leqslant x \leqslant 1-t$

$$f_t(x) := \frac{1}{t}\int_x^{x+t} f(s)\,ds \tag{2.14}$$

and extend it to $[1-t, t]$ by $f_t(x) := f_t(1-t)$.

Then $f_t(x) \in M_0\,[0,1] \cap \dot{W}^1_p$ and therefore

$$K(f, t, L_p \cap M_0, \dot{W}^1_p \cap M_0) \leqslant \|f - f_t\|_{L_p} + t\|f_t'\|_{L_p}. \tag{2.15}$$

Since $K(f, t, L_p, \dot{W}^1_p) \approx \omega_1(f,t)_{L_p}$ (see, e.g. [9]) it remains to prove that the right-hand side of (2.15) is bounded by $\omega_1(f,t)_{L_p}$.

We have

$$\|f - f_t\|_{L_p}^p = \int_0^{1-t} \left| \frac{1}{t} \int_0^t [f(x) - f(x+s)]\, ds \right|^p dx$$
$$+ \int_{1-t}^1 |f(x) - f_t(1-t)|^p\, dx := I_1 + I_2.$$

By Hölder's inequality we get

$$I_1 \leqslant \frac{1}{t} \int_0^t ds \int_0^{1-t} |f(x) - f(x+t)|^p\, dx \leqslant \omega_1^p(f,t)_{L_p}$$

Applying now the Lebesgue inequality (cf. [9, p. 30]) and then a Whitney-type inequality (see the general case in [2]) we get

$$I_2 \leqslant CE_0(f;[1-t,1])_{L_p}^p \leqslant C\omega_1^p(f,t)_{L_p}. \tag{2.16}$$

Here and below C denotes an absolute constant, not necessarily the same (in this case, independent of f and t). Putting these inequalities together we get

$$\|f - f_t\|_{L_p} \leqslant C\omega_1(f,t)_{L_p}.$$

To estimate the second term of (2.15) we note that for a.e. $x \in [0,1]$

$$f_t'(x) = \begin{cases} \frac{1}{t}\Delta_t f(x) & \text{if } 0 \leqslant x \leqslant 1-t, \\ 0 & \text{if } 1-t < x \leqslant 1. \end{cases}$$

Therefore

$$t\|f_t'\|_{L_p} \leqslant \omega_1(f,t)_{L_p},$$

whence the result follows.

(ii) We have to prove that

$$(L_p, \dot{B}_p^{\alpha,\infty} \cap M_0)_{\theta,q} (L_p, \dot{B}_p^{\alpha,\infty})_{\theta,q} \cap M_0,$$

where $\alpha > 1/p$. This embedding is a consequence of the following:

Proposition 2.1. *Let $f \in L_p[0,1] \cap M_0$, where $1 \leqslant p \leqslant \infty$ and let $r > 1/p$ be an integer. Then for every $t \in (0,1]$ there is a function $h_t \in W_p^r \cap M_0$*

such that

$$\|f-h_t\|_{L_p(0,1)}+t^r\|h_t^{(r)}\|_{L_p(0,1)}$$

$$\leqslant C\left\{t^{1/p}\int_0^t \frac{\omega_r(f,s)_{L_p}}{s^{1+1/p}}\,ds+t^r\int_t^1 \frac{\omega_r(f,s)_{L_p}}{s^{r+1}}\,ds\right\}. \tag{2.17}$$

Proof. We can suppose that $\int_0^t \frac{\omega_r(f,u)_{L_p}}{u^{1+1/p}}\,du<\infty$ which implies that f coincides a.e. with a continuous function on $[0,1]$. It suffices to construct h_t with $t=\frac{1}{n}, n\in\mathbb{N}$. Consider first the case $n\geqslant 2$. Introduce a subdivision π_n of the interval $[0,1]$ into the subintervals $I_{n,k}$ where $I_{n,k}:=[x_{n,k-1},x_{n,k})$, $k=1,\dots,n-1$ and $I_{n,n}:=[x_{n,n-1},x_{n,n}]$, $x_{n,0}=0$ and $x_{n,n}=1$ satisfying the condition that there are constants $\alpha,\beta>0$ independent of k and n, such that

$$\frac{\alpha}{n}\leqslant|I_{n,k}|\leqslant\frac{\beta}{n}. \tag{2.18}$$

Consider the enlargements of $I_{n,k}$ defined as follows: $\tilde{I}_{n,k}:=[a_{n,k},x_{n,k}]$, $k=2,\dots,n$, where $a_{n,k}:=\frac{x_{n,k-1}+x_{n,k-2}}{2}$, and $\tilde{I}_{n,1}=[0,x_{n,1}]$.

Let $P_{n,k}\in\mathscr{P}_{r-1}(\tilde{I}_{n,k})$ be the polynomial of best uniform approximation of $f\in M_0(\tilde{I}_{n,k})$. Then $\tilde{P}_{n,k}:=P_{n,k}-\min\{\inf_{\tilde{I}_{n,k}}P_{n,k},0\}$ is a nonnegative polynomial providing near best approximation on $\tilde{I}_{n,k}$:

$$\|f-\tilde{P}_{n,k}\|_{L_\infty(\tilde{I}_{n,k})}\leqslant 2\|f-P_{n,k}\|_{L_\infty(\tilde{I}_{n,k})}=2E_{r-1}(f,\tilde{I}_{n,k})_{L_\infty(0,1)}$$
$$\leqslant 2\gamma\omega_r\left(f,\frac{|\tilde{I}_{n,k}|}{r}\right)_{L_\infty(\tilde{I}_{n,k})}. \tag{2.19}$$

The last inequality follows from Whitney's theorem and γ does not depend on f and $\tilde{I}_{n,k}$.

We now construct the required sequence $\{h_{n,k}\}_{1\leqslant k\leqslant n}$.

Set for $x\in[0,x_{n,1}]$

$$h_{n,1}(x):=\tilde{P}_{n,1}(x)(\geqslant 0).$$

Suppose now that $h_{n,j}$ have already been defined for $j=1,\dots,k$, where $k<n$. We define $h_{n,k+1}$ by

$$h_{n,k+1}(x):=\begin{cases} h_{n,k}(x) & \text{if } 0\leqslant x\leqslant a_{n,k+1},\\ \tilde{P}_{n,k}(x)[1-\rho_{n,k}(x)]+\tilde{P}_{n,k+1}(x)\rho_{n,k}(x) & \text{if } a_{n,k+1}<x\leqslant x_{n,k},\\ \tilde{P}_{n,k+1}(x) & \text{if } x_{n,k}<x\leqslant x_{n,k+1},\end{cases}$$

where

$$\rho_{n,k}(x) := \beta_{n,k} \int_{a_{n,k+1}}^{x} (t - a_{n,k+1})^r (x_{n,k} - t)^r \, dt$$

and the normalizing coefficient $\beta_{n,k}$ is determined by the condition

$$\rho_{n,k}(x_{n,k}) = 1.$$

Then $0 \leqslant \rho_{n,k}(x) \leqslant 1$ for $x \in [a_{n,k+1}, x_{n,k}]$.

It is the matter of checking definition that $\{h_{n,k}\}$ has the following properties:

(a) $h_{n,k} \in C^r[0, x_{n,k}] \cap M_0$;
(b) $h_{n,k} = \tilde{P}_{n,k}$ for $x_{n,k-1} < x \leqslant x_{n,k}$;
(c) $\|f - h_{n,k}\|^p_{L_p[0,a_{n,k+1}]} \leqslant \frac{(2C(\alpha,\beta)\gamma)^p}{n}\{2\sum_{1\leqslant j\leqslant k-1} \omega_r^p(f, \frac{|\tilde{I}_{n,j}|}{r})_{L_\infty(\tilde{I}_{n,j})} + \omega_r^p (f, \frac{|\tilde{I}_{n,k}|}{r})_{L_\infty(\tilde{I}_{n,k})}\}$.

Thus, the sequence $\{h_{n,k}\}_{1\leqslant k\leqslant n}$ has yet to be constructed. Now define the desired function h_t with $t = \frac{1}{n}$ by

$$h_t := h_{n,n}.$$

Then $h_t \in C^r[0,1] \cap M_0$ by (a). Moreover,

$$\|f - h_t\|^p_{L_p[0,a_{n,n+1}]} \leqslant \frac{(2C(\alpha,\beta)\gamma)^p}{n} \left\{ 2 \sum_{1\leqslant j\leqslant n-1} \omega_r^p\left(f, \frac{|\tilde{I}_{n,j}|}{r}\right)_{L_\infty(\tilde{I}_{n,j})} + \omega_r^p\left(f, \frac{|\tilde{I}_{n,n}|}{r}\right)_{L_\infty(\tilde{I}_{n,n})} \right\}.$$

Since $h_t := h_{n,n} := \tilde{P}_{n,n}(x)$ if $x \in [a_{n,n+1}, 1] \subset \tilde{I}_{n,n}$, we also have

$$\|f - h_t\|^p_{L_p[a_{n,n+1},1]} \leqslant \frac{(2C(\alpha,\beta)\gamma)^p}{3n} \omega_r^p\left(f, \frac{|\tilde{I}_{n,n}|}{r}\right)_{L_\infty(\tilde{I}_{n,n})}.$$

These two inequalities lead to the estimate

$$\|f - h_t\|^p_{L_p[0,1]} \leqslant \frac{2(2C(\alpha,\beta)\gamma)^p}{n} \sum_{1\leqslant j\leqslant n} \omega_r^p\left(f, \frac{|\tilde{I}_{n,j}|}{r}\right)_{L_\infty(\tilde{I}_{n,j})}.$$

We now recall the embedding estimate already used at the beginning of the proof

$$\omega_r(f,t)_{L_\infty(a,b)} \leqslant C(r)\int_0^t \frac{\omega_r(f,s)_{L_p(a,b)}}{s^{1+1/p}}\,ds.$$

Using this together with an integral form of Minkowski's inequality, we then obtain

$$\|f-h_t\|_{L_p[0,1]} \leqslant C(\alpha,\beta,\gamma)n^{-1/p}\int_0^{\frac{C(\alpha,\beta)}{nr}} s^{-1-1/p}\left\{\sum_{1\leqslant j\leqslant n}\omega_r^p(f,s)_{L_p(\tilde I_{n,j})}\right\}^{1/p} ds. \tag{2.20}$$

To estimate the sum in (2.20) we apply the following result (see [3, Theorem 4, Section 2] or [5, Theorem 1]):

$$\omega_r(f,s)_{L_p(a,b)} \approx \sup_\pi\left\{\sum_{I\in\pi}E_r(f,I)^p_{L_p}\right\}^{1/p} \qquad \left(s<\frac{b-a}{r}\right), \tag{2.21}$$

where the supremum is taken over all families π of nonoverlapping subintervals of $[a,b]$ of length s and the constants of equivalence depend only on r and p. From here it follows that for all $s\in(0,\frac{C(\alpha,\beta)}{nr})$ we have

$$\left\{\sum_{k=1}^n \omega_r^p(f,s)_{L_p(\tilde I_{n,k})}\right\}^{1/p} \leqslant C\left\{\sum_{k=1}^n\sum_{I\in\pi_{n,k}}E_r(f,I)^p_{L_p}\right\}^{1/p}. \tag{2.22}$$

Here $\pi_{n,k}$ is a family from (2.21) (with $[a,b]:=\tilde I_{n,k}$) for which the supremum is attained within ϵ. Because of the choice of intervals $\tilde I_{n,k}$, any two intervals $I\in\pi_{n,k}$ and $I'\in\pi_{n,k+2}$ are nonoverlapping. Dividing the family $\bigcup_{k=1}^n\pi_{n,k}$ into two subfamilies of nonoverlapping intervals and applying (2.21) to each of these families we estimate the right-hand side of (2.22) by $C_1\omega_r(f,s)_{L_p(0,1)}$ with C_1 depending only on r.

From this estimate and (2.20) it follows that $\|f-h_t\|_{L_p[0,1]}$ is dominated by $C\Psi(\frac{C(\alpha,\beta)}{nr})$, where $\Psi(t):=t^{1/p}\int_0^t\frac{\omega_r(f,s)_{L_p(0,1)}}{s^{1+1/p}}ds,\ \ t>0.$

Since $\Psi(\frac{C(\alpha,\beta)}{nr})\leqslant C(\alpha,\beta,r)\Psi(\frac{1}{n})$ this leads to the inequality

$$\|f-h_t\|_{L_p[0,1]}\leqslant C\Psi(t) \qquad \left(t=\frac{1}{n}\right). \tag{2.23}$$

To obtain (2.17) it now remains to estimate the Sobolev norm of $h_t(:=h_{n,n})$. Since $h_{n,n}$ is a spline of degree $\leqslant 3r$ with almost uniformly distributed knots, we can apply Markov's inequality (see, e.g. [9, p. 103]) to estimate

$(h_{2t} - h_t)^{(r)}$ with $t = \frac{1}{n}$. So we have

$$\|(h_{2t} - h_t)^{(r)}\|_{L_p[0,1]} \leqslant \left\{ \sum_{k=1}^{n} \|(h_{2t} - h_t)^{(r)}\|^p_{L_p(\tilde{I}_{n,k})} \right\}^{1/p} \leqslant C t^{-r} \Psi(2t). \qquad (2.24)$$

Choose $m \in \mathbb{N}$ such that $2^{-m-1} < t = \frac{1}{n} \leqslant 2^{-m}$.
Then

$$\|h_t^{(r)}\|_{L_p[0,1]} \leqslant \sum_{j=1}^{m} \|(h_{2^j t} - h_{2^{j-1} t})^{(r)}\|_{L_p[0,1]} + \|h^{(r)}_{2^m t}\|_{L_p[0,1]}. \qquad (2.25)$$

Since $h_{2^m t} \coloneqq h_1 \coloneqq h_{1,1}$ is, by construction, a polynomial of degree $\leqslant r-1$, $h^{(r)}_{2^m t} = 0$ and from (2.24) and (2.25) we deduce that

$$\|h_t^{(r)}\|_{L_p[0,1]} \leqslant C \sum_{j=1}^{m} (2^j t)^{(1/p-r)} \int_0^{2^j t} \frac{\omega_r(f,s)_{L_p[0,1]}}{s^{1+1/p}} \, ds$$

$$= C \sum_{j=1}^{m} \int_{2^{j-1} t}^{2^j t} \left(\int_0^{2^j t} \frac{\omega_r(f,s)_{L_p[0,1]}}{s^{1+1/p}} \, ds \right) \frac{du}{u^{r-1/p+1}}. \qquad (2.26)$$

By elementary properties of ω_r (see, e.g. [9, p. 45]) the preceding expression is dominated by $\{t^{-r+1/p} \int_0^t \frac{\omega_r(f,s)_{L_p[0,1]}}{s^{1+1/p}} \, ds + \int_t^1 \frac{\omega_r(f,s)_{L_p[0,1]}}{s^{r+1}} \, ds\}$.

Together with (2.23) this proves inequality (2.17) in case $t = 1/n$ and $n \geqslant 2$.

It remains to consider the case $t = 1$. Let P be a polynomial of degree $\leqslant r-1$ for which

$$\|f - P\|_{L_\infty[0,1]} = E_{r-1}(f;[0,1])_{L_\infty}.$$

Set $h_1 \coloneqq P - \min\{\inf_{[0,1]} P, 0\}$. Exactly as in the argument just after (2.19) we get

$$\|f - h_1\|_{L_p[0,1]} + \|h_1^{(r)}\|_{L_p[0,1]} = \|f - h_1\|_{L_p[0,1]}$$

$$\leqslant \|f - h_1\|_{L_\infty[0,1]} \leqslant C \int_0^1 \frac{\omega_r(f,s)_{L_p[0,1]}}{s^{1+1/p}} \, ds \qquad (2.27)$$

The proof of the proposition is complete. ∎

COROLLARY 2.2. *If* $\theta > \frac{1}{rp}$, *then*

$$M_0 \cap (L_p, \dot{W}^r_p \cap M_0)_{\theta,q} = (L_p, \dot{W}^r_p \cap M_0)_{\theta,q} = (L_p, \dot{W}^r_p)_{\theta,q} \cap M_0.$$

Proof. For each $f \in (L_p, \dot{W}^r_p)_{\theta,q} \cap M_0 \subset L_p \cap M_0$ we have from Proposition 2.1 that

$$\begin{aligned}\tilde{K}(t^r) &:= K(f, t^r, L_p, \dot{W}^r_p \cap M_0) \\ &\leqslant C\left[t^{1/p}\int_0^t \frac{\omega_r(f,u)_{L_p}}{u^{1+1/p}}\,du + t^r\int_t^1 \frac{\omega_r(f,u)_{L_p}}{u^{r+1}}\,du\right].\end{aligned}$$

Applying the $L_q^{r\theta}(\frac{dt}{t})$-norm to both sides and then using Hardy's inequalities, we have for $\theta > \frac{1}{rp}$

$$\begin{aligned}\|\tilde{K}\|_{L_q^{r\theta}(dt/t)} &\leqslant C\left[\left\|t^{1/p}\int_0^t \frac{\omega_r(f,u)_{L_p}}{u^{1+1/p}}\,du\right\|_{L_q^{r\theta}(dt/t)} + \left\|t^r\int_t^1 \frac{\omega_r(f,u)_{L_p}}{u^{r+1}}\,du\right\|_{L_q^{r\theta}(dt/t)}\right] \\ &\leqslant C\|\omega_r(f;\cdot)_{L_p}\|_{L_q^{r\theta}(dt/t)}. \end{aligned} \tag{2.28}$$

This last expression is equivalent to the $L_q^{r\theta}(dt/t)$ norm of the function $K(f, t^r, L_p, \dot{W}^r_p)$, i.e. we have shown that $\|\tilde{K}\|_{L_q^{r\theta}(dt/t)} \leqslant C|f|_{(L_p,\dot{W}^r_p)_{\theta,q}}$. Since the reverse inequality is obvious, this shows that $M_0 \cap (L_p, \dot{W}^r_p \cap M_0)_{\theta,q} = (L_p, \dot{W}^r_p)_{\theta,q} \cap M_0$. Finally, it is easy to check that any function in $(L_p, \dot{W}^r_p \cap M_0)_{\theta,q}$ must be nonnegative a.e. and so $(L_p, \dot{W}^r_p \cap M_0)_{\theta,q} = M_0 \cap (L_p, \dot{W}^r_p \cap M_0)_{\theta,q}$ completing the proof of the corollary. ∎

To complete the proof of Theorem A it suffices to show that for $f \in M_0$

$$\|K(f,\cdot; L_p, \dot{B}^{\alpha,\infty}_p \cap M_0)\|_{L_q^\eta(dt/t)} \leqslant C\|K(f,\cdot; L_p, \dot{B}^{\alpha,\infty}_p)\|_{L_q^\eta(dt/t)}, \tag{2.29}$$

provided

$$\frac{1}{p\alpha} < \eta < 1 \qquad \text{and} \qquad 1 \leqslant q \leqslant \infty. \tag{2.30}$$

Given any $\alpha > 1/p$ we set $r = [\alpha] + 1$ and $\theta = \frac{\alpha}{r}$. Then

$$(L_p, \dot{W}^r_p)_{\theta,\infty} = \dot{B}^{\alpha,\infty}_p$$

and $\theta > \frac{1}{rp}$. So we can apply Corollary 2.2 to obtain that for each $f \in M_0$

$$K(f, t, L_p, \dot{B}_p^{\alpha,\infty} \cap M_0) \leqslant CK(f, t, L_p, (L_p, \dot{W}_p^r)_{\theta,\infty} \cap M_0)$$

$$\leqslant CK(f, t, L_p, (L_p, \dot{W}_p^r \cap M_0)_{\theta,\infty}). \qquad (2.31)$$

To estimate this last expression we apply a following variant of Holmstedt's formula (for proof see [21]).

Proposition 2.3. *Suppose that for fixed $\theta \in (0, 1)$ and $q \in [1, \infty]$ the isomorphism*

$$Q \cap (X_0, X_1 \cap Q)_{\theta,q} = (X_0, X_1)_{\theta,q} \cap Q$$

holds. Then

$$K(f, t^\theta, X_0, (X_0, X_1 \cap Q)_{\theta,q}) \approx t^\theta \left(\int_t^\infty [s^{-\theta} K(f, s, X_0, X_1 \cap Q)]^q \frac{ds}{s} \right)^{1/q}.$$

Combining Corollary 2.2 and this proposition we get

$$\|K(f, t, L_p, \dot{B}_p^{\alpha,\infty} \cap M_0)\|_{L_q^\eta(dt/t)} \leqslant C \left\| t \sup_{s \geqslant t^{1/\theta}} \frac{K(f, s, L_p, \dot{W}_p^r \cap M_0)}{s^\theta} \right\|_{L_q^\eta(dt/t)}. \qquad (2.32)$$

To estimate the right-hand side of (2.32) we note that, since the function $K(s) := K(f, s, L_p, \dot{W}_p^r \cap M_0)$ is increasing, it satisfies

$$\frac{K(s)}{s^\theta} \leqslant (q\theta)^{1/q} \left(\int_s^\infty \left(\frac{K(u)}{u^\theta} \right)^q \frac{du}{u} \right)^{1/q}$$

for every $s > 0$. Then taking the supremum over $s \geqslant t^{1/\theta}$ and changing the order of integration we get

$$\left\| t \sup_{s \geqslant t^{1/\theta}} \frac{K(f, s)}{s^\theta} \right\|_{L_q^\eta(dt/t)}^q \leqslant (q\theta) \int_0^\infty \left(t^q \int_{t^{1/\theta}}^\infty \left(\frac{K(u)}{u^\theta} \right)^q \frac{du}{u} \right) \frac{dt}{t^{q\eta+1}}$$

$$= \frac{\theta}{(1-\eta)} \|K(f, \cdot, L_p, \dot{W}_p^r \cap M_0)\|_{L_q^{\theta\eta}(dt/t)}^q. \qquad (2.33)$$

So we have proved that for $f \in M_0$

$$\|K(f, \cdot, L_p, \dot{B}_p^{\alpha,\infty} \cap M_0)\|_{L_q^{\eta}(dt/t)} \leqslant C\|K(f, \cdot, L_p, \dot{W}^r_p \cap M_0)\|_{L_q^{\theta\eta}(dt/t)}. \tag{2.34}$$

Since, by our condition (2.30) on η, we have $\theta\eta = \frac{\alpha\eta}{r} > \frac{\alpha}{r} \cdot \frac{1}{p\alpha} = \frac{1}{rp}$. Applying now Corollary 2.2 to show that the right-hand side of the preceding inequality is majorized by

$$\|K(f, \cdot, L_p, \dot{W}^r_p)\|_{L_q^{\theta\eta}(dt/t)} = \|f\|_{(L_p, \dot{W}^r_p)_{\theta\eta,q}}. \tag{2.35}$$

By the "end point" version of Holmstedt's theorem (see, e.g., formula (3.16)) of [13, p. 186] we have

$$(L_p, \dot{W}^r_p)_{\theta\eta,q} = (L_p, (L_p, \dot{W}^r_p)_{\theta,\infty})_{\eta,q} = (L_p, \dot{B}_p^{\alpha,\infty})_{\eta,q}. \tag{2.36}$$

Thus the right-hand side of (2.35) up to a constant does not exceed

$$\|f\|_{(L_p,\dot{B}_p^{\alpha,\infty})_{\eta,q}} = \|K(\cdot, f, L_p, \dot{B}_p^{\alpha,\infty})\|_{L_q^{\eta}(dt/t)}.$$

Putting together (2.34), (2.35) and the last inequality we obtain required inequality (2.29). The proof of Theorem A is complete. ∎

3. PROOF OF THEOREM B

(a) We apply part (a) of Proposition 1.8 in the case $X := L_p[0,1]$, $Y := \dot{W}^1_p[0,1]$, $Q := M_0$ and $A_n = S_n^1$, $n \in \mathbb{N}$. It is clear that $\{S_n^1\}$ is an approximation family. We have to verify that the assumptions of this result hold true in our situation. In the other words, we should prove

(i) $M_0[0,1]$ has the IP_+ with respect to $(L_p, \dot{W}^1_p)$
(ii) If $f \in \dot{W}^1_p(0,1) \cap M_0$, then

$$\sigma_{n,1}^{M_0}(f; L_p) \leqslant Cn^{-1}\|f'\|_{L_p}, \qquad n \in \mathbb{N}. \tag{3.37}$$

But (i) has already been proved in part (i) of Theorem A. To prove the second statement define $s_n \in S_n^1 \cap M_0$ by

$$s_n(x) := f_{\Delta_{i,n}}\left(:= \frac{1}{|\Delta_{i,n}|}\int_{\Delta_{i,n}} f\,dx\right), \quad x \in \Delta_{i,n},\ i = 1, 2, \ldots, n,$$

and $s_n(1) := \lim_{x\to 1-0} s_n(x)$. Then $s_n \geqslant 0$, since $f \geqslant 0$. Therefore,

$$\sigma_{n,1}^{M_0}(f; L_p) \leqslant \|f - s_n\|_{L_p(0,1)} \leqslant \left\{ \sum_{i=0}^{n-1} \int_{\Delta_{i,n}} |f(x) - f_{\Delta_{i,n}}|^p \, dx \right\}^{1/p}.$$

Using inequality (2.16) in the proof of Theorem A(i) for the case of the interval Δ and $f \in \dot{W}_p^1(\Delta)$ we have

$$\int_{\Delta} |f(x) - f_{\Delta}|^p \, dx \leqslant C\omega_1^p(f, |\Delta|, \Delta)_p \leqslant C|\Delta|^p \|f'\|_{L_p(\Delta)}^p.$$

Summing these inequalities with $\Delta := \Delta_{i,n}$, $i = 1, 2, \ldots, n$, we get (3.37).

Thus, in our case Proposition 1.8(a) implies for $f \in L_p(0,1) \cap M_0$:

$$\sigma_{n,1}^{M_0}(f; L_p) \leqslant CK(f, n^{-1}, L_p, \dot{W}_p^1), \;\; n \in \mathbb{N}.$$

It remains to note that the right-hand side is equivalent to $\omega_1(f, n^{-1})_{L_p}$.

(b) First we prove that

$$(L_p, \dot{B}_p^{r,\infty})_{\frac{\alpha}{r},\infty} \cap M_0 \subset E_{\alpha,\infty}^{M_0}(\{S_n^{k,l}\}, L_p), \tag{3.38}$$

where r is the smallest integer $> \alpha (> \frac{1}{p})$. To do this we have to prove the following abstract Jackson's inequality: if $f \in \dot{B}_p^{r,\infty}(0,1) \cap M_0$, $k := 3r + 4, l := r + 1$ then

$$\sigma_{n,k.l}^{M_0}(f; L_p) \leqslant Cn^{-r} |f|_{\dot{B}_p^{r,\infty}(0,1)}, \;\; n \in \mathbb{N}. \tag{3.39}$$

Then (3.38) will follow from here and Proposition 1.8.

To prove (3.39) make use of the function $h_{n,n}$ constructed in the proof of Proposition 2.1, but with $r+1$ instead of r. We also take in this case the subdivision $\pi_n := \{\Delta_{i,n}\}_{i=1}^n$ from Definition 1.10 for the construction of $h_{n,n}$. That is, let $I_{n,i}$ from the proof of Proposition 2.1 coincides with $\Delta_{i,n}$. Then it follows that the intervals $\tilde{I}_{n,i}$ which arise in this proof must each be finite unions of intervals from the finer subdivision $\{\Delta_{i,2n}\}_{i=1}^n$. Consequently, the restriction of the function $h_{n,n}$ constructed in the proof of Proposition 2.1 to each interval $\Delta_{i,2n}$ is a polynomial of degree not exceeding $3r+4$. Furthermore, $h_{n,n} \in C^{r+1}[0,1] \cap M_0$. Thus $h_{n,n} \in S_{2n}^{k,l}$ with $k := 3r+4$, $l := r+1$.

Therefore, for $f \in L_p(0,1) \cap M_0$ we get

$$\sigma_{2n,k.l}^{M_0}(f; L_p) \leqslant \|f - h_{n,n}\|_{L_p[0,1]}.$$

By inequality (2.23) we also have

$$\begin{aligned}\|f-h_{n,n}\|_{L_p[0,1]} &\leqslant Cn^{-1/p}\int_0^{n^{-1}}\frac{\omega_{r+1}(f,s)_{L_p(0,1)}}{s^{1+1/p}}\,ds\\ &\leqslant Cn^{-1/p}\left(\sup_{s>0}\frac{\omega_{r+1}(f,s)_{L_p(0,1)}}{s^r}\right)\int_0^{n^{-1}} s^{r-1-1/p}\,ds\\ &\leqslant Cn^{-r}|f|_{\dot{B}_p^{r,\infty}(0,1)}. \end{aligned}\tag{3.40}$$

Combining this and the previous inequality we get (3.39) for all even integers. If n is odd then $\sigma^{M_0}_{2n+1,k.l}(f;L_p)\leqslant\sigma^{M_0}_{2n,k.l}(f;L_p)$.

From (3.39) and Proposition 1.8 we have for $f\in L_p\cap M_0$

$$\sup_{n\in\mathbb{N}}\sigma^{M_0}_{n,k.l}(f;L_p)n^{\alpha}\leqslant C\sup_{n\in\mathbb{N}}K(f,n^{-r},L_p,\dot{B}_p^{r,\infty}\cap M_0)n^{\alpha}=C|f|_{(L_p,\dot{B}_p^{r,\infty}\cap M_0)_{\theta,\infty}}.$$

Since $\theta:=\frac{\alpha}{r}>\frac{1}{rp}$ it follows from Theorem A(ii) that $M_0[0,1]$ has the restricted WIP$_+$ with respect to $(L_p,\dot{B}_p^{r,\infty})$ for $\theta:=\frac{\alpha}{r}$, i.e. for $f\in L_p\cap M_0$

$$|f|_{(L_p,\dot{B}_p^{r,\infty}\cap M_0)_{\theta,\infty}}\leqslant C|f|_{(L_p,\dot{B}_p^{r,\infty})_{\theta,\infty}}\leqslant C\|f\|_{\dot{B}_p^{\alpha,\infty}}.$$

Embedding (3.38) is proved. ∎

To prove the opposite statement we need the following version of Bernstein's inequality. It can be obtain by a modification of proof for Lemma 2 in [5, p. 156], see also [14, 17].

Lemma 3.1. *Let $k,l,0<l<k$, be integers and let $p\in[1,\infty]$. Then there exists a constant $\gamma=\gamma(k,l,p)$ such that*

$$\omega_{k+1}(S,t)_{L_p[0,1]}\leqslant\gamma(nt)^{l+1+1/p}\|S\|_{L_p[0,1]}\tag{3.41}$$

holds for all spline functions $S\in S_n^{k,l}[0,1]$.

Let now $f\in L_p[0,1]$, and there exists a sequence of splines $S_n\in S_n^{k,l}[0,1]$, $k=3l+1$, such that

$$\sup_{n\in\mathbb{N}}n^{\alpha}\|f-S_n\|_n=C_f<\infty.\tag{3.42}$$

By Lemma 3.1 each $\phi\in S_n^{k,l}[0,1]$ satisfies

$$\omega_{k+1}(\phi,t)_{L_p[0,1]}\leqslant\gamma(nt)^{l+1+1/p}\|\phi\|_{L_p[0,1]}$$

for each $n \in \mathbb{N}$. This immediately implies the following Bernstein-type inequality:

$$|\phi|_{\dot{B}^{l+1+1/p,\infty}_{p,(k+1)}[0,1]} \leqslant Cn^{l+1+1/p} \|\phi\|_{L_p[0,1]} \qquad \text{for all } \phi \in S_n^{k,l}[0,1], \tag{3.43}$$

where the constant C depends only on l. We now apply part (ii) of Theorem 5.1 of [9, p. 216] in the case where $X = L_p[0,1]$, $Y = B^{l+1+1/p,\infty}_{p,(k+1)}[0,1)$, and $\Phi_n = S_n^{k,l}[0,1]$. Since Y is semi-normed we have $\mu(Y) = 1$ in (5.6) of [9, p. 216] and (3.43) corresponds to condition (5.5) of [9, p. 216] with the parameter r appearing there replaced by $\rho = l + 1 + 1/p$. Thus we obtain condition (5.8) of [9, p. 217], which in our case can be written in the form

$$K(f, n^{-\rho}, L_p, \dot{B}^{\rho,\infty}_{p,(k+1)}) \leqslant Cn^{-\rho} \sum_{j=1}^{n} j^{\rho-1} \sigma^{M_0}_{j,k,l}(f; L_p).$$

Estimating the right-hand side by (3.42) we have

$$K(f, n^{-\rho}, L_p, \dot{B}^{\rho,\infty}_{p,(k+1)}) \leqslant C' n^{-\alpha} \qquad \text{for all } n \in \mathbb{N}.$$

Then, by a standard argument we obtain, in turn, that for some $C'' = C(f, \alpha, p, l)$

$$K(f, t, L_p, \dot{B}^{\rho,\infty}_{p,(k+1)}) \leqslant C'' t^{\alpha/\rho} \qquad \text{for all } t \in (0,1].$$

Consequently, $f \in (L_p, \dot{B}^{\rho,\infty}_{p,(k+1)})_{\alpha/\rho,\infty} = (L_p, (L_p, \dot{W}^{k+1}_p)_{\rho/(k+1),\infty})_{\alpha/\rho,\infty}$. By an "endpoint" version of the reiteration theorem (see e.g. [1]) this last space is $(L_p, \dot{W}^{k+1}_p)_{\alpha/(k+1),\infty}$. In other words, we have shown that the $(k+1)$th modulus of smoothness of f satisfies

$$\sup_{t \in [0,1]} \frac{\omega_{k+1}(f,t)_{L_p}}{t^\alpha} < \infty$$

or, equivalently, $f \in \dot{B}^{\alpha,\infty}_{p,(k+1)}[0,1)$. Hence, $f \in B^{\alpha,\infty}_p[0,1)$. The proof of Theorem B is complete. ∎

Remark. (a) It is possible to decrease the numbers k, l in the proof of part (b). Namely, the following inequality holds for $f \in \dot{B}^{r,\infty}_p(0,1) \cap M_0$:

$$\sigma^{M_0}_{n,r}(f; L_p) \leqslant Cn^{-\alpha} |f|_{\dot{B}^{\alpha,\infty}_p(0,1)}, \quad n \in \mathbb{N}.$$

Here $r > \alpha > 1/p$.

(b) Results similar to Theorem B and that of the previous two remarks hold for the cone M_1 of nonnegative nondecreasing functions on $[0, 1]$. In this case the IP_+ of M_1 with respect to $(L_p, \dot{W}_p^2)$ plays a basic role.

It is an interesting open problem to determine whether the analog of Theorem B is valid for the general cone M_k of k-monotone functions with $k \geqslant 2$.

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